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Systems of generalized quasivariational inclusions problems with applications to variational analysis and optimization problems

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Abstract In this paper, we study an existence theorem of systems of generalized quasivariational inclusions problem. By this result, we establish the existence theorems of solutions of systems of generalized equations, systems of generalized vector quasiequilibrium problem, collective variational fixed point, systems of generalized quasiloose saddle point, systems of minimax theorem, mathematical program with systems of variational inclusions constraints, mathematical program with systems of equilibrium constraints and systems of bilevel problem and semi-infinite problem with systems of equilibrium problem constraints.

Keywords Variational inclusion · Saddle point · Minimax theorem · Mathematical program with systems of equilibrium constraints · Bilevel problem · Semi-infinite problem · Upper (lower) semicontinuous multivalued map

1 Introduction

In 1979, Robinson [30] studied the following variational inclusions problem:
For each $x \in R^n$, find $y \in R^m$ such that

$$0 \in g(x, y) + Q(x, y) \tag{1a}$$

where $g\colon R^n \times R^m \to R^p$ is a single valued function and $Q\colon R^n \times R^m \multimap R^p$ is a multivalued map. It is known that (1a) covers a vast of variational system important in applications. For example, model (1a) can be reduced to a parametric variational

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inequality: Find $y \in \Omega$ such that

$$\langle q(x,y), u-y\rangle \geq 0 \quad \text{for all} u \in \Omega.$$

Since then, various types of variational inclusions problems have been extended and generalized by Hassouni and Moudafi [14], Adly [1], Ahmad and Ansan [2], Chang [9], Ding [10], Huang [16], Ahmad et al. [3], etc. Very recently, Morduckhovich [29] study equilibrium problem with various inclusions constraint of model (1a). For further references on variational inclusions, one can refer to Mordukhovich [28,29] and references therein.

Let X be a topological vector space, let $K \subseteq X$ be a nonempty set and f: $K \times K \to \mathbb{R}$ be a bifunction with $f(x,x) = 0$ for all $x \in X$. Then the equilibrium problem (EP) Blum et al. [8] is to find $\bar{x} \in X$ such that $f(\bar{x}, y) \geq 0$ for all $y \in K$.

It is known that equilibrium problem and systems of equilibrium problem contain variational inequalities problem, optimization problems, fixed point problem, complementary problems, saddle point problems and Nash equilibrium as special cases. For detail, one can refer to Ansari et al. [4], Blum and Oettli [8], Lin [18–20], Lin et al. [21–25] and references therein.

Generalized semi-infinite problems are programs of the type

$$\text{SIP:} \qquad \min_x f(x) \quad \text{s.t.} \;\; \phi(x,t) \geq 0 \quad \forall t \in H(x)\,, \tag{1b}$$

where $H(x) \subset \mathbb{R}^m$ is the index set defined by a set-valued mapping H: $\mathbb{R}^n \multimap \mathbb{R}^m$. Bilevel problems are of the form

$$\begin{aligned} \text{BL:} \qquad & \min_{x,y} f(x,y) \quad \text{s.t.} \qquad g(x,y) \geq 0, \\ & \text{and } y \text{ is a solution of } Q(x)\text{:} \quad \min_t F(x,t) \quad \text{s.t.} \quad t \in H(x). \end{aligned} \tag{2}$$

We also consider mathematical programs with equilibrium constraints

$$\begin{aligned} \text{MPEC:} \qquad & \min_{x,y} f(x,y) \quad \text{s.t.} \;\; g(x,y) \geq 0\,, \quad y \in H(x) \\ & \qquad\qquad \text{and} \quad \phi(x,y,t) \geq 0 \qquad \forall t \in H(x). \end{aligned} \tag{3}$$

These programs represent three important classes of optimization problems which have been investigated in a large number of papers and books (see Refs. [6,7,13, 27] and the references therein). As usual in linear and nonlinear optimization, these papers mainly deal with optimality conditions and numerical methods to solve the problems. Typically the existence of a feasible point is tacitly assumed (see Lin and Still [24]). Recently, Lin et al. [21,22,24], Lin [18,19] investigate under which assumptions the existence of feasible points can be assumed in advanced.

One can easily see that the above problems also have many relations with the following problems.

Let I be an index set. For each $i \in I$, let Z_i be a real topological vector space (in short, t.v.s.), X_i and Y_i be nonempty closed convex subsets of locally convex t.v.s. E_i and V_i, respectively. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let A_i: $X \times Y \multimap X_i$, T_i: $X \multimap Y_i$, G_i: $X \times Y \times Y_i \multimap Z_i$ and $C_i, D_i : X \multimap Z_i$ be multivalued maps. In this paper, we study the following type of systems of variational inclusions problems:

(SVIP) Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$; (i.e. $0 \in G_i(\bar{x}, \bar{y}, T_i(\bar{x}))$) for all $i \in I$.

(SVIP) contains the following problems as special cases:

(i) If H_i: $X \times Y \multimap Z_i$, Q_i: $X \times Y \times Y_i \multimap Z_i$ and $G_i(x,y,v_i) = H_i(x,y) + Q_i(x,y,v_i)$, then (SVIP) will be reduced to the following problem:
Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in H_i(\bar{x},\bar{y}) + Q_i(\bar{x},\bar{y},v_i)$ for all $v_i \in T_i(\bar{x})$. That is $0 \in H_i(\bar{x},\bar{y}) + Q_i(\bar{x},\bar{y},T_i(\bar{x}))$ for all $i \in I$.

(ii) If $G_i(x,y,v_i) = -D_i(x) + Q_i(x,y,v_i)$, then (SVIP) will be reduced to the systems of equilibrium problem:
Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $G_i(\bar{x},\bar{y},v_i) \cap D_i(\bar{x}) \neq \emptyset$ for all $v_i \in T_i(\bar{x})$ for all $i \in I$.

(iii) If $G_i(x,y,v_i) = [Z_i \setminus (-\text{int}D_i(x))] + Q_i(x,y,v_i)$, then (SVIP) will be reduced to the systems of equilibrium problem:
Find $(\bar{x},\bar{y}) \in X \times Y$, $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, $Q_i(\bar{x},\bar{y},v_i) \nsubseteq -\text{int}D_i(\bar{x})$ for all $v_i \in T_i(\bar{x})$ and all $i \in I$.

(iv) If H: $X \times Y \multimap X$ and Q_i: $X \times Y \times Y_i \multimap X$ and if $H_i(x,y) = \{-x\}$ for each $i \in I$, then (SVIP) will be reduced to the systems of variational fixed point problem:
Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$ and $\bar{x} \in Q_i(\bar{x},\bar{y},v_i)$ for all $v_i \in T_i(\bar{x})$ and all $i \in I$. That is $\bar{x} \in Q_i(\bar{x},\bar{y},T_i(\bar{x}))$ for all $i \in I$.

(v) If G_i: $X \times Y \times Y_i \to Z_i$ is a single valued function, then (SVIP) will be reduced to the systems of generalized quasivariational equation problem:
Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 = G_i(\bar{x},\bar{y},v_i)$ for all $v_i \in T_i(\bar{x})$ for all $i \in I$.
(SVIP) also have many applications.

(vi) If S_i: $X \multimap X_i$, F_i: $X \times Y \times X_i \multimap Z_i$ and let $A_i(x,y) = \{w_i \in S_i(x)\colon\ 0 \in F_i(w,y,u_i)$ for all $u_i \in S_i(x)\}$, then (SVIP) will be reduced to the following problem:
Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$0 \in F_i(\bar{x},\bar{y},u_i) \quad \text{for all } u_i \in S_i(\bar{x})$$

and

$$0 \in G_i(\bar{x},\bar{y},v_i) \quad \text{for all } v_i \in T_i(\bar{x}), \quad \text{for all } i \in I.$$

(vi) contains (vii), (viii), (ix) and (x) as special cases.

(vii) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$F_i(\bar{x},\bar{y},u_i) \cap C_i(\bar{x}) \neq \emptyset \quad \text{for all } u_i \in S_i(\bar{x})$$

and

$$G_i(\bar{x},\bar{y},v_i) \cap D_i(\bar{x}) \neq \emptyset \quad \text{for all } v_i \in T_i(\bar{x}), \quad \text{for all } i \in I.$$

(viii) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$F_i(\bar{x},\bar{y},u_i) \not\subset -\text{int}C_i(\bar{x}) \quad \text{for all } u_i \in S_i(\bar{x})$$

and

$$G_i(\bar{x},\bar{y},v_i) \cap D_i(\bar{x}) \neq \emptyset \quad \text{for all } v_i \in T_i(\bar{x}) \quad \text{and} \quad \text{for all } i \in I.$$

(ix) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$F_i(\bar{x},\bar{y},u_i) \not\subset -\text{int}C_i(\bar{x}) \quad \text{for all } u_i \in S_i(\bar{x})$$

and

$$G_i(\bar{x},\bar{y},v_i) \not\subset -\operatorname{int}D_i(\bar{x}) \neq \emptyset \quad \text{for all } v_i \in T_i(\bar{x}) \quad \text{and} \quad \text{for all } i \in I.$$

(x) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$ and

$$F_i(\bar{x},\bar{y},u_i) \cap C_i(\bar{x}) \neq \emptyset \quad \text{for all } u_i \in S_i(\bar{x})$$

and

$$G_i(\bar{x},\bar{y},v_i) \nsubseteq -\operatorname{int}D_i(\bar{x}) \text{ for all } v_i \in T_i(\bar{x}) \text{ and for all } i \in I.$$

As applications of our results, we study the mathematical program with equilibrium constraint, bilevel problem and semi-infinite problems, our approach are different from Fukushima and Pang [13], Bard [6], Luo et al. [27] and Lin et al. [18,22,24].

If F_i: $X \times Y \multimap Z_i$ and h: $X \times Y \multimap Z_0$, where Z_0 is a real t.v.s. ordered by proper closed conve cone C_0 in Z_0, (SVIP) can be applied to studied the following problem:

(xi) mathematical program with systems of variational inclusions constraints: $\min_{(x,y)} h(x,y)$, $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$ such that $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \subseteq C_i(x)$ and $0 \in G_i(x,y,v_i)$ for all $v_i \in T_i(x)$ and for all $i \in I$.

(xii) mathematical program with systems of equilibrium constraints:
(SMPEC 1) $\min_{(x,y)} h(x,y)$, $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$ such that $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \subseteq D_i(x)$ and $G_i(x,y,v_i) \cap D_i(x) \neq \emptyset$ for all $v_i \in T_i(x)$ and for all $i \in I$.
or
(SMPEC 2) $\min_{(x,y)} h(x,y)$, $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$ such that $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \subseteq -\operatorname{int} D_i(x)$ and $G_i(x,y,v_i) \nsubseteq -\operatorname{int} D_i(x)$ for all $v_i \in T_i(x)$ and for all $i \in I$.

If $Z_i = \mathbb{R}$ for all $i \in I$, $Z_0 = \mathbb{R}$ and $C_i(x) = \mathbb{R}^+ = [0,\infty)$ for all $x \in X$ and $i \in I$, then (SMPEC 1) and (SMPEC 2) will be reduced to the following problem:

(xiii) $\min_{(x,y)} h(x,y)$, $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$ such that $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \geq 0$ and $G_i(x,y,v_i) \geq 0$ for all $v_i \in T_i(x)$ and for all $i \in I$.

(ixv) If Q_i: $X \times Y_i \to \mathbb{R}$ and $G_i(x,y,v_i) = Q_i(x,v_i) - Q_i(x,y_i)$, then (SMPEC 1) and (SMPEC 2) will reduce to the systems of bilevel problem:
$\min_{(x,y)} h(x,y)$, $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$ such that $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \geq 0$ and y_i is a solution to the problem $\min_{v_i \in T_i(x)} Q_i(x,v_i)$ for all $i \in I$.

(xv) For the special cases of systems of bilevel problem is the semi-infinite problem with systems of equilibrium constraints:
$\min_{(x,y)} h(x,y)$, $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$ such that $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \geq 0$ and $G_i(x,v_i) \geq 0$ for all $v_i \in T_i(x)$ and for all $i \in I$.

(xvi) In (i), if $H_i(x,y) = (-\infty,0]$, B_i: $X \to W_i^*$, η_i: $Y \times Y_i \to Y_i$, and $G_i(x,y,v_i) = \langle B_i(x), \eta(y,v_i)\rangle$, where W_i^* is the dual space of W_i, then (i) will reduce to the following mixed variational-like inequality problem:
Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $\langle B_i(\bar{x}), \eta(\bar{y},v_i)\rangle \geq 0$ for all $v_i \in T_i(\bar{x})$ and for all $i \in I$.

In this paper, we first study the existence theorem of systems of generalized quasivariational inclusions, from which we study the existence theorems of systems of generalized quasiequilibrium problems, systems of variational fixed point problems, systems of generalized quasivariational equations. As applications, we study the existence theorems of two family of variational inclusions, systems of simultaneous quasiequilibrium problems. We also study the existence theorems of mathematical program with systems of generalized quasivariational inclusions constraints, mathematical program with systems of equilibrium constraints, systems of bilevel problems and semi-infinite problem with systems of equilibrium constraints. Our results on system of generalized quasiequilibrium problems are different from Lin [18,23] and Lin et al. [22,23,25]. Our results on systems on simultaneous quasiequilibrium problems are different from Ansari et al. [4] and Chang [19]. Our results on mathematical problem with systems of equilibrium constraints are different from Bard [6], Birbil et al. [7], Fukushima and Pang [13], Lin et al. [22,24], Lin [18], and Luo et al. [27].

2 Preliminaries

Let X and Y be topological spaces (in short t.s.), $T\colon X \multimap Y$ be a multivalued map. T is said to be upper semicontinuous (in short u.s.c.) (respectively lower semicontinuous (in short l.s.c.) at $x \in X$, if for every open set U in Y with $T(x) \subseteq U$ (resp. $T(x) \cap U \neq \emptyset$) there exists an open neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (resp. $T(x') \cap U \neq \emptyset$) for all $x' \in T(x)$; T is said to be u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; T is continuous at x if T is both u.s.c. and l.s.c. at x; T is compact if there exists a compact set K such that $T(X) \subseteq K$; T is closed if $GrT = \{(x, y) \in X \times Y\colon y \in T(x), x \in X\}$ is a closed set in $X \times Y$.

Let Z be a real t.v.s., D a proper convex cone in Z. A point $\bar{y} \in A$ is called a vector minimal point of A if for any $y \in A$, $y - \bar{y} \notin -D \setminus \{0\}$. The set of vector minimal point of A is denoted by $\mathrm{Min}_D A$.

The following Lemmas and theorems are need in this paper.

Lemma 2.1 ([31]) *Let X and Y be topological spaces, $T\colon X \multimap Y$ be a multivalued map. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$ in X converges to x, there exists a net $\{y_\alpha\}_{\alpha\in\Lambda}$, $y_\alpha \in T(x_\alpha)$ for all $\alpha \in A$ with $y_\alpha \to y$.*

Lemma 2.2 ([26]) *Let Z be a Hausdorff t.v.s., C be a closed convex cone in Z. If A is a nonempty compact subset of Z, then $\mathrm{Min}_C A \neq \emptyset$.*

Theorem 2.1 ([5]) *Let X and Y be Hausdorff topological spaces, $T\colon X \multimap Y$ be a multivalued map.*

(i) *If T is an u.s.c. multivalued map with closed values, then T is closed.*
(ii) *If Y is a compact space and T is closed, then T is u.s.c.*
(iii) *If X is compact and T is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

Definition 2.1 ([11]) Let E be a vector space and $X \subseteq E$ an arbitrary subset. A multivalued map $F\colon X \multimap E$ is said to be a KKM map provided

$$\mathrm{conv}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$$

for each finite subset $\{x_1, \ldots, x_n\} \subseteq X$, where $\mathrm{conv}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

The basic property of KKM map is given in Theorem 2.2.

Definition 2.2 Let X be a nonempty convex subset of a vector space E, Y be a nonempty convex subset of a vector space H and Z be a real t.v.s. Let $F: Y \multimap Z$ and $C: X \multimap Z$ be multivalued maps such that for each $x \in X$, $C(x)$ is a closed convex cone.

(i) F is $C(x)$-quasiconvex if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, either

$$F(y_1) \subseteq F(\lambda y_1 + (1-\lambda)y_2) + C(x)$$

or

$$F(y_2) \subseteq F(\lambda y_1 + (1-\lambda)y_2) + C(x)$$

(ii) F is $C(x)$-quasiconvex-like if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, either

$$F(\lambda y_1 + (1-\lambda)y_2) \subseteq F(y_1) - C(x)$$

or

$$F(\lambda y_1 + (1-\lambda)y_2) \subseteq F(y_2) - C(x)$$

(iii) F is $\{0\}$-quasiconvex-like if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, either

$$F(\lambda y_1 + (1-\lambda)y_2) \subseteq F(y_1)$$

or

$$F(\lambda y_1 + (1-\lambda)y_2) \subseteq F(y_2)$$

(iv) F is affine if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$,

$$F(\lambda y_1 + (1-\lambda)y_2) = \lambda F(x, y_1) + (1-\lambda)F(x, y_2).$$

(v) F is concave if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, we have

$$\lambda F(y_1) + (1-\lambda)F(y_2) \subseteq F(\lambda y_1 + (1-\lambda)y_2).$$

(vi) F is $\{0\}$-quasiconvex if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, either

$$F(y_1) \subseteq F(\lambda y_1 + (1-\lambda)y_2)$$

or

$$F(y_2) \subseteq F(\lambda y_1 + (1-\lambda)y_2)$$

Theorem 2.2 ([11]) *Let E be a t.v.s, X be an arbitrary subset of E, and $F: X \multimap E$ a KKM map. If $G(x)$ is closed for each $x \in X$ and if $G(x_0)$ is compact for some $x_0 \in X$, then $\cap\{G(x): x \in X\} \neq \emptyset$.*

Theorem 2.3 ([15]) *Let X be a convex subset of a locally convex t.v.s. and D be a nonempty compact subset of X, $T: X \multimap D$ be an u.s.c. multivalued map such that for each $x \in X$, $T(x)$ is a nonempty closed convex subset of D. Then there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.*

Theorem 2.4 ([25]) *Let E_1, E_2, and Z be Hausdorff t.v.s., X and Y be nonempty subsets of E_1 and E_2, respectively. Let $F: X \times Y \multimap Z$, $S: X \multimap Z$ be multivalued maps, and let $T: X \multimap Y$ be defined by $T(x) = \cup_{y \in S(x)} F(x, y) = F(x, S(x))$.*

(a) *If both S and F are l.s.c., then T is l.s.c. on X.*
(b) *If both S and F are u.s.c. multivalued maps with compact values, then T is an u.s.c. multivalued map with compact values.*

Throughout this paper, we assume that all topological spaces are Hausdorff.

3 Existence results for a solution of system of generalized quasivariational inclusions problems

The following theorem is the main result of this paper.

Theorem 3.1 *Let I be any index set. For each $i \in I$, let X_i be a nonempty convex subset of a locally convex t.v.s. E_i, Z_i be a t.v.s., and Y_i be a nonempty convex subset of a t.v.s. W_i. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, suppose that*

(i) *A_i: $X \times Y \multimap X_i$ is a compact u.s.c. multivalued map with nonempty closed convex values;*
(ii) *T_i: $X \multimap Y_i$ is a compact continuous multivalued map with nonempty closed convex values;*
(iii) *G_i: $X \times Y \times Y_i \multimap Z_i$ is a closed multivalued map with nonempty values and for each $(x, v_i) \in X \times Y_i$, $y \multimap G_i(x, y, v_i)$ is concave or $\{0\}$-quasiconvex;*
(iv) *for each $(x, y) \in X \times Y$ with $y = (y_i)_{i\in I}$, $v_i \multimap G_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and $0 \in G_i(x, y, y_i)$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x}_i)_{i\in I}$ and $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

Proof For each $i \in I$, let H_i: $X \multimap T_i(X)$ be defined by

$$H_i(x) = \{y_i \in T_i(x)\colon 0 \in G_i(x, y, v_i) \text{ for all } v_i \in T_i(x), \text{ and for } y = (y_i)_{i\in I}\}.$$

Then $H_i(x)$ is nonempty for each $x \in X$ and $i \in I$. Indeed, for each $i \in I$ and $x \in X$, let $Q_i(x)$: $T_i(x) \multimap T_i(x)$ be defined by

$$Q_i(x)(v_i) = \{y_i \in T_i(x) : 0 \in G_i(x, y, v_i)\}.$$

Then $Q_i(x)$ is a KKM map. Indeed, suppose that $Q_i(x)$ is not a KKM map, then there exists a finite set $\{v_i^1, v_i^2, \ldots, v_i^n\}$ in $T_i(x)$ such that $co\{v_i^1, \ldots, v_i^n\} \nsubseteq \bigcup_{k=1}^n Q_i(x)(v_i^k)$. Hence there exists $v_i^\lambda = \lambda_1 v_i^1 + \cdots + \lambda_n v_i^n \in co\{v_i^1, v_i^2, \ldots, v_i^n\}$ such that $v_i^\lambda \notin Q_i(x)(v_i^k)$ for all $k = 1, 2, \ldots, n$, where $\lambda_j \geq 0$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n \lambda_j = 1$. Since $T_i(x)$ is convex, $v_i^\lambda \in co\{v_i^1, v_i^2, \ldots, v_i^n\} \subseteq T_i(x)$. But $v_i^\lambda \notin Q_i(x)(v_i^k)$ for all $k = 1, 2, \ldots, n$, we see that $0 \notin G_i(x, v^\lambda, v_i^k)$ for all $k = 1, 2, \ldots, n$ where $v^\lambda = (v_i^\lambda)_{i\in I}$. By (iv), there exists $1 \leq j \leq n$ such that

$$0 \in G_i(x, v^\lambda, v_i^\lambda) \subseteq G_i(x, v^\lambda, v_i^j).$$

This leads to a contradiction. Therefore $Q_i(x)$ is a KKM map. For each $i \in I$ and $v_i \in Y_i$, $Q_i(x)(v_i)$ is a closed set. Indeed, if $y_i \in \overline{Q_i(x)(v_i)}$, then there exists a net $\{y_i^\alpha\}$ in $Q_i(x)(v_i)$ such that $y_i^\alpha \to y_i$. Let $y^\alpha = (y_i^\alpha)_{i\in I}$ and $y = (y_i)_{i\in I}$. One has $y_i^\alpha \in T_i(x)$ and $0 \in G_i(x, y^\alpha, v_i)$. Since $T_i(x)$ is a closed set and G_i is closed, $y_i \in T_i(x)$ and $0 \in G_i(x, y, v_i)$. This shows that $y_i \in Q_i(x)(v_i)$ and $Q_i(x)(v_i)$ is a closed set for each $i \in I$ and $v_i \in Y_i$. By (ii), $\overline{T_i(X)}$ is compact and $Q_i(x)(v_i) \subseteq \overline{T_i(X)}$, we see that $Q_i(x)(v_i)$ is compact for each $v_i \in Y_i$ and $i \in I$. Then by KKM Theorem, $\bigcap_{v_i\in T_i(x)} Q_i(x)(v_i) \neq \emptyset$. Let $y_i \in \bigcap_{v_i\in T_i(x)} Q_i(x)(v_i)$, then $0 \in G_i(x, y, v_i)$ for all $v_i \in T_i(x)$ and $H_i(x)$ is nonempty for each $x \in X$ and $i \in I$. H_i is closed for each $i \in I$. Indeed, if $(x, y_i) \in \overline{GrH_i}$, then there exists a net $(x^\alpha, y_i^\alpha) \in GrH_i$ such that $(x^\alpha, y_i^\alpha) \to (x, y_i)$. Let $y^\alpha = (y_i^\alpha)_{i\in I}$ and $y = (y_i)_{i\in I}$. One has $y_i^\alpha \in T_i(x^\alpha)$ and $0 \in G_i(x^\alpha, y^\alpha, v_i)$ for all $v_i \in T_i(x^\alpha)$. By (ii) and Theorem 2.1, T_i is closed and $y_i \in T_i(x)$. Let $v_i \in T_i(x)$. Since T_i is l.s.c., it follows from Lemma 2.1 that there exists a net $\{v_i^\alpha\}$ such that $v_i^\alpha \in T_i(x^\alpha)$ and $v_i^\alpha \to v_i$.

We have $0 \in G_i(x^\alpha, y^\alpha, v_i^\alpha)$. Since G_i is closed, $0 \in G_i(x, y, v_i)$ for all $v_i \in T_i(x)$. This shows that $(x, y_i) \in GrH_i$ and GrH_i is a closed set. Therefore H_i is closed for each $i \in I$. But $H_i(X) \subseteq \overline{T_i(X)}$ and $\overline{T_i(X)}$ is compact, it follows from Theorem 2.1 that H_i: $X \multimap Y_i$ is a compact u.s.c. multivalued map. $H_i(x)$ is convex for each $x \in X$ and $i \in I$. Indeed, let $y_i^1, y_i^2 \in H_i(x)$ and $\lambda \in [0,1]$. Let $y^1 = (y_i^1)_{i\in I}$ and $y^2 = (y_i^2)_{i\in I}$, then $y_i^1, y_i^2 \in T_i(x)$, $0 \in G_i(x, y^1, v_i)$ and $0 \in G_i(x, y^2, v_i)$ for all $v_i \in T_i(x)$. Therefore $\lambda y_i^1 + (1-\lambda)y_i^2 \in T_i(x)$. By (iii),

$$0 \in G_i(x, \lambda y^1 + (1-\lambda)y^2, v_i)$$

for all $v_i \in T_i(x)$. This shows that $\lambda y_i^1 + (1-\lambda)y_i^2 \in H_i(x)$ and $H_i(x)$ is convex. Since H_i is closed, it is easy to see that $H_i(x)$ is a closed set for each $x \in X$. Let Q: $X \times Y \multimap X \times Y$ be defined by

$$Q(x, y) = \left[\prod_{i\in I} A_i(x, y)\right] \times \left[\prod_{i\in I} H_i(x)\right].$$

It follows from Lemma 3 [12] that $\prod_{i\in I} A_i(x, y)$, $\prod_{i\in I} H_i(x)$ and Q are compact u.s.c. multivalued maps with nonempty closed convex values. Then by Himmelberg fixed point Theorem that there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $(\bar{x}, \bar{y}) \in Q(\bar{x}, \bar{y})$. Hence for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. □

For the particular cases of Theorem 3.1, we have the following Theorems and Corollaries.

Theorem 3.2 *In Theorem 3.1, if conditions* (iii) *and* (iv) *are replaced by* $(\text{iii})_1$ *and* $(\text{iv})_1$ *respectively, where*

- $(\text{iii})_1$ *$H_i : X \times Y \multimap Z_i$ is a closed multivalued map with nonempty values and Q_i: $X \times Y \times Y_i \multimap Z_i$ is an u.s.c. multivalued map with nonempty compact values;*
- $(\text{iv})_1$ *for each $(x, v_i) \in X \times Y_i$, $y \multimap H_i(x, y)$ and $y \multimap Q_i(x, y, v_i)$ are concave or $\{0\}$-quasiconvex; for each $(x, y) \in X \times Y$, $v_i \multimap Q_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and $0 \in H_i(x, y) + Q_i(x, y, y_i)$, where $y = (y_i)_{i\in I}$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x}_i)_{i\in I}$ and $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in H_i(\bar{x}, \bar{y}) + Q_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

Proof For each $i \in I$, let G_i: $X \times Y \times Y_i \multimap Z_i$ be defined by

$$G_i(x, y, v_i) = H_i(x, y) + Q_i(x, y, v_i).$$

Then G_i is a closed multivalued map. Indeed, if $(x, y, v_i, w_i) \in \overline{GrG_i}$, then there exists a net $\{(x^\alpha, y^\alpha, v_i^\alpha, w_i^\alpha)\}_{\alpha\in\Lambda}$ in GrG_i such that $(x^\alpha, y^\alpha, v_i^\alpha, w_i^\alpha) \to (x, y, v_i, w_i)$. One has $w_i^\alpha \in G_i(x^\alpha, y^\alpha, v_i^\alpha) = H_i(x^\alpha, y^\alpha) + Q_i(x^\alpha, y^\alpha, v_i^\alpha)$. There exist $u_i^\alpha \in H_i(x^\alpha, y^\alpha)$ and $z_i^\alpha \in Q_i(x^\alpha, y^\alpha, v_i^\alpha)$ such that $w_i^\alpha = u_i^\alpha + z_i^\alpha$. Let $K = \{x^\alpha\}_{\alpha\in\Lambda} \cup \{x\}$, $L = \{y^\alpha\}_{\alpha\in\Lambda} \cup \{y\}$ and $M_i = \{v_i^\alpha\}_{\alpha\in\Lambda} \cup \{v_i\}$. Then K is a compact set in X, L and M_i are compact sets in Y and Y_i respectively. By $(\text{iii})_1$ and Theorem 2.1 that $Q_i(K \times L \times M_i)$ is a compact set. There exists a subnet $\{z_i^{\alpha_\lambda}\}$ of $\{z_i^\alpha\}$ such that $z_i^{\alpha_\lambda} \to t_i$. Since Q_i is an u.s.c. multivalued map with nonempty closed values, it follows from Theorem 2.1 that Q_i is closed and $t_i \in Q_i(x, y_i, v_i)$. But $w_i^\alpha - z_i^\alpha = u_i^\alpha \in H_i(x^\alpha, y^\alpha)$ and $w_i^\alpha - z_i^\alpha \to w_i - t_i$. By assumption, H_i is closed. We have $u_i \in H_i(x, y)$ and $w_i = t_i + u_i \in H_i(x, y) + Q_i(x, y, v_i)$. Hence $(x, y, v_i, w_i) \in GrG_i$ and G_i is closed. It is easy to see that conditions (iii) and (iv) of Theorem 3.1 hold.

Then by Theorem 3.1 that there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in G_i(\bar{x}, \bar{y}, v_i) = H_i(\bar{x}, \bar{y}) + Q_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. □

Remark 3.1 Theorem 3.2 implies that there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x})_{i\in I}$, $\bar{y} = (\bar{y})_{i\in I}$, such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in H_i(\bar{x}, \bar{y}) + Q_i(\bar{x}, \bar{y}, T_i(\bar{x}))$;

For the special case of Theorem 3.2, we establish the following existence theorems of systems of generalized vector equilibrium problem.

Corollary 3.1 *In Theorem 3.1, if conditions* (iii) *and* (iv) *are replaced by* $(\mathrm{iii})_2$ *and* $(\mathrm{iv})_2$ *respectively, where*

- $(\mathrm{iii})_2$ *C_i: $X \multimap Z_i$ is a closed multivalued map and Q_i: $X \times Y \times Y_i \multimap Z_i$ is an u.s.c. multivalued map with nonempty compact values;*
- $(\mathrm{iv})_2$ *for each $(x, v_i) \in X \times Y_i$, $y \multimap Q_i(x, y, v_i)$ is concave or $\{0\}$-quasiconvex, and for each $(x, y) \in X \times Y$, $v_i \multimap Q_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and $Q_i(x, y, y_i) \cap C_i(x) \neq \emptyset$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x}_i)_{i\in I}$ and $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $Q_i(\bar{x}, \bar{y}, v_i) \cap C_i(\bar{x}) \neq \emptyset$ for all $v_i \in T_i(\bar{x})$.

Proof For each $i \in I$, let H_i: $X \times Y \multimap Z_i$ be defined by $H_i(x, y) = -C_i(x)$ for all $x \in X$. Then H_i is a closed multivalued map with nonempty values. For each $(x, y) \in X \times Y$, $v_i \multimap Q_i(x, y, v_i) - C_i(x)$ is $\{0\}$-quasiconvex-like. Since $G_i(x, y, y_i) \cap C_i(x) \neq \emptyset$ for each $(x, y) \in X \times Y$, $0 \in -C_i(x) + Q_i(x, y, y_i)$ for each $(x, y) \in X \times Y$ with $y = (y_i)_{i\in I}$.

Then by Theorem 3.2 that there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} = (\bar{x}_i)_{i\in I}$, $\bar{y} = (\bar{y}_i)_{i\in I}$, such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in -C_i(\bar{x}) + Q_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. Hence $Q_i(\bar{x}, \bar{y}, v_i) \cap C_i(\bar{x}) \neq \emptyset$ for all $v_i \in T_i(\bar{x})$. □

Following the same argument as in Corollary 3.1, we have the following Corollary.

Corollary 3.2 *In Corollary 3.1, if conditions* $(\mathrm{iii})_2$ *and* $(\mathrm{iv})_2$ *are replaced by* $(\mathrm{iii})_3$ *and* $(\mathrm{iv})_3$, *respectively, where*

- $(\mathrm{iii})_3$ *C_i: $X \multimap Z_i$ is a multivalued map such that* $\mathrm{int}C_i(x)$ *is nonempty for each $x \in X$ and W_i: $X \multimap Z_i$ defined by $W_i(x) = Z_i \setminus (-\mathrm{int}C_i(x))$ is an u.s.c. multivalued map;*
- $(\mathrm{iv})_3$ *Q_i: $X \times Y \times Y_i \multimap Z_i$ is an u.s.c. multivalued map with nonempty compact values. For each $(x, v_i) \in X \times Y_i$, $y \multimap Q_i(x, y, v_i)$ is concave or $\{0\}$-quasiconvex, and for each $(x, y) \in X \times Y$, $v_i \multimap Q_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and $Q_i(x, y, y_i) \nsubseteq -\mathrm{int}C_i(x)$, where $y = (y_i)_{\cdot i\in I}$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x}_i)_{i\in I}$ and $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $Q_i(\bar{x}, \bar{y}, v_i) \nsubseteq -\mathrm{int}C_i(\bar{x})$ for all $v_i \in T_i(\bar{x})$.

Proof It follows from Theorem 3.2 that there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} = (\bar{x}_i)_{i\in I}$, $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in -W_i(\bar{x}) + Q_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. From this, we obtain $Q_i(\bar{x}, \bar{y}, v_i) \nsubseteq -\mathrm{int}C_i(\bar{x})$ for all $v_i \in T_i(\bar{x})$. □

Remark 3.2

(i) In Corollaries 3.1 and 3.2, we do not assume that $C_i(x)$ is a convex cone for each $x \in X$.

(ii) Corollary 3.1 is true if the condition "for each $(x, y) \in X \times Y$, $v_i \multimap Q_i(x, y, v_i)$ is 0-quasiconvex-like" is replaced by "for each $(x, y) \in X \times Y$, $v_i \multimap Q_i(x, y, v_i)$ is $C_i(x)$-quasiconvex-like and $C_i(x)$ is a nonempty convex cone."

Proof Let $G_i(x, y, v_i) = -C_i(x) + Q_i(x, y, v_i)$. Let $v_i^{(1)}$, $v_i^{(2)} \in Y_i$ and $\lambda \in (0, 1)$. By assumption, either

$$Q_i(x, y, \lambda v_i^{(1)} + (1 - \lambda)v_i^{(2)}) \subseteq Q_i(x, y, v_i^{(1)}) - C_i(x).$$

or

$$Q_i(x, y, \lambda v_i^{(1)} + (1 - \lambda)v_i^{(2)}) \subseteq Q_i(x, y, v_i^{(2)}) - C_i(x).$$

Since $C_i(x)$ is a convex cone, either

$$\begin{aligned} G_i(x, y, \lambda v_i^{(1)} + (1 - \lambda)v_i^{(2)}) &= Q_i(x, y, \lambda v_i^{(1)} + (1 - \lambda)v_i^{(2)}) - C_i(x) \\ &\subseteq Q_i(x, y, v_i^{(1)}) - C_i(x) - C_i(x) = Q_i(x, y, v_i^{(1)}) - C_i(x) \\ &= G_i(x, y, v_i^{(1)}). \end{aligned}$$

or

$$\begin{aligned} G_i(x, y, \lambda v_i^{(1)} + (1 - \lambda)v_i^{(2)}) &= Q_i(x, y, \lambda v_i^{(1)} + (1 - \lambda)v_i^{(2)}) - C_i(x) \\ &\subseteq Q_i(x, y, v_i^{(2)}) - C_i(x) = G_i(x, y, v_i^{(2)}). \end{aligned}$$

This shows that for each $(x, y_i) \in X \times Y_i$, $v_i \multimap G_i(x, y, v_i)$ is 0-quasiconvex-like. Then by Theorem 3.1, Corollary 3.1 is true if the condition "for each $(x, y_i) \in X \times Y_i$, $v_i \multimap Q_i(x, y, v_i)$ is 0-quasiconvex-like" is replaced by "$C_i(x)$ is a convex cone and $v_i \multimap Q_i(x, y, v_i)$ is $C_i(x)$-quasiconvex-like."

Remark 3.3 Theorem 3.2 is true if conditions $(\text{iii})_1$ and $(\text{iv})_1$ are replaced by $(\text{iii})_4$ and $(\text{iv})_4$, respectively, where

$(\text{iii})_4$ H_i: $X \times Y \to Z_i$ is a continuous function and Q_i: $X \times Y \times Y_i \multimap Z_i$ is an u.s.c. multivalued map with nonempty compact values;

$(\text{iv})_4$ for each $(x, v_i) \in X \times Y_i$, $y \to H_i(x, y)$ and $y \multimap Q_i(x, y, v_i)$ are concave or $\{0\}$-quasiconvex, and for each $(x, y) \in X \times Y$, $v_i \multimap Q_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and $0 \in H_i(x, y) + Q_i(x, y, y_i)$.

Corollary 3.3 *In Theorem 3.1, if conditions* (iii) *and* (iv) *are replaced by* $(\text{iii})_5$ *and* $(\text{iv})_5$*, respectively, where*

$(\text{iii})_5$ *Q_i, P_i: $X \times Y \times Y_i \multimap Z_i$ are u.s.c. multivalued map with nonempty compact values;*

$(\text{iv})_5$ *for each $(x, v_i) \in X \times Y_i$, $y \multimap P_i(x, y, v_i)$ and $y_i \multimap Q_i(x, y, v_i)$ are concave or $\{0\}$-quasiconvex; for each $(x, y) \in X \times Y_i$, $v_i \multimap Q_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and $0 \in P_i(x, y, T_i(x)) + Q_i(x, y, y_i)$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x}_i)_{i \in I}$ and $\bar{y} = (\bar{y}_i)_{i \in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in P_i(\bar{x}, \bar{y}, w_i) + Q_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$ and for all $w_i \in T_i(\bar{x})$.

Proof For each $i \in I$, let H_i: $X \times Y \multimap Z_i$ be defined by $H_i(x,y) = P_i(x,y,T_i(x))$. Since both T_i and P_i are u.s.c. multivalued maps with nonempty compact values, it follows from Theorem 2.4 that H_i: $X \times Y \multimap Z_i$ is an u.s.c. multivalued map with nonempty compact values. Again, by Theorem 2.1 that H_i: $X \times Y \multimap Z_i$ is a closed multivalued map. Then Corollary 3.3 follows from Theorem 3.2. □

Theorem 3.3 *In Theorem 3.1, if condition* (i) *is replaced by* (i′)*, where*

(i′) $S_i : X \multimap X_i$ *is a compact continuous multivalued map with nonempty closed convex values.*

And we assume further that

(v) F_i: $X \times Y \times X_i \multimap Z_i$ *is a closed multivalued map with nonempty values and for each* $(y,u_i) \in Y \times X_i$, $w \multimap F_i(w,y,u_i)$ *is concave or* $\{0\}$*-quasiconvex;*

(vi) *for each* $(x,y) \in X \times Y$, $u_i \multimap F_i(x,y,u_i)$ *is* $\{0\}$*-quasiconvex-like and* $0 \in F_i(x,y,x_i)$ *for* $x = (x_i)_{i\in I}$.

Then there exists $(\bar{x},\bar{y}) \in X \times Y$ *with* $\bar{x} = (\bar{x}_i)_{i\in I}$ *and* $\bar{y} = (\bar{y}_i)_{i\in I}$ *such that for each* $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $0 \in F_i(\bar{x},\bar{y},u_i)$ *for all* $u_i \in S_i(\bar{x})$ *and* $0 \in G_i(\bar{x},\bar{y},v_i)$ *for all* $v_i \in T_i(\bar{x})$.

Proof For each $i \in I$, let A_i: $X \times Y \multimap X_i$ be defined by
$A_i(x,y) = \{w_i \in S_i(x) : 0 \in F_i(w,y,u_i)$ for all $u_i \in S_i(x)$, for $w = (w_i)_{i\in I}\}$.

Then we follow the same argument as in Theorem 3.1, we can prove that A_i: $X \times Y \multimap X_i$ is a compact u.s.c. multivalued map with nonempty closed convex values. Then it follows from Theorem 3.1 that there exists $(\bar{x},\bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x}_i)_{i\in I}$ and $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in G_i(\bar{x},\bar{y},v_i)$ for all $v_i \in T_i(\bar{x})$. Therefore $\bar{x}_i \in S_i(\bar{x})$, $0 \in F_i(\bar{x},\bar{y},u_i)$ for all $u_i \in S_i(\bar{x})$. □

For the another special cases of Theorem 3.2, we have the following Corollaries.

Corollary 3.4 *In Theorem 3.1, if conditions* (iii) *and* (iv) *are replaced by* $(\text{iii})_6$ *and* $(\text{iv})_6$*, respectively, where*

$(\text{iii})_6$ $Q_i : X \times Y \times Y_i \multimap X$ *is an u.s.c. multivalued map with nonempty compact values;*

$(\text{iv})_6$ *for each* $(x,v_i) \in X \times Y_i$, $y \multimap Q_i(x,y,v_i)$ *is concave or* $\{0\}$*-quasiconvex, and for each* $(x,y) \in X \times Y$, $v_i \multimap Q_i(x,y,v_i)$ *is* $\{0\}$*-quasiconvex-like and* $x \in Q_i(x,y,y_i)$.

Then there exists $(\bar{x},\bar{y}) \in X \times Y$ *with* $\bar{x} = (\bar{x}_i)_{i\in I}$ *and* $\bar{y} = (\bar{y}_i)_{i\in I}$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ *and* $\bar{x} \in Q_i(\bar{x},\bar{y},v_i)$ *for all* $v_i \in T_i(\bar{x})$.

Proof For each $i \in I$, let $H_i : X \times Y \multimap X_i$ be defined by $H_i(x,y) = \{-x\}$ for all $(x,y) \in X \times Y$. Then H_i is a closed multivalued map with nonempty convex values and Corollary 3.4 follows from Theorem 3.2. □

The following Corollary is an existence theorem of systems of variational equations.

Corollary 3.5 *In Theorem 3.1 and Remark 3.1, if conditions* (iii) *and* (iv) *are replaced by* $(\text{iii})_7$ *and* $(\text{iv})_7$*, respectively, where*

$(\text{iii})_7$ G_i: $X \times Y \times Y_i \to Z_i$ *is a continuous function and for each* $(x,v_i) \in X \times Y_i$, $y \to G_i(x,y,v_i)$ *is concave or* $\{0\}$*-quasiconvex;*

(iv)$_7$ *for each* $(x,y) \in X \times Y$, $v_i \to G_i(x,y,v_i)$ *is* $\{0\}$*-quasiconvex and* $G_i(x,y,y_i) = 0$, *where* $y = (y_i)_{i \in I}$.

Then there exists $(\bar{x},\bar{y}) \in X \times Y$ *with* $\bar{x} = (\bar{x}_i)_{i \in I}$ *and* $\bar{y} = (\bar{y}_i)_{i \in I}$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ *and* $0 = G_i(\bar{x},\bar{y},v_i)$ *for all* $v_i \in T_i(\bar{x})$.

Corollary 3.6 *In Theorem 3.1, suppose conditions* (i) *and* (ii) *and suppose that*

(a) W_i^* *is the dual space of* W_i, B_i: $X \to W_i^*$, η_i: $Y \times Y_i \to Y_i$;
(b) *For each* $(x,v_i) \in X \times Y_i$, $y \to \eta(y,v_i)$ *is affine, for each* $(x,y) \in X \times Y$, $v_i \to \langle B_i(x), \eta_i(y,v_i)\rangle$ *is* $\{0\}$*-quasiconvex and* $\eta(y,y_i) = 0$ *for all* $y = (y_i)_{i \in I}$.

Then there exist $\bar{x} = (\bar{x}_i)_{i \in I} \in X$, $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ *and* $\langle B_i(\bar{x}), \eta_i(\bar{y},v_i)\rangle \geq 0$ *for all* $v_i \in T_i(\bar{x})$.

Proof Let $H_i(x,y) = (-\infty, 0]$ and $Q_i(x,y,v_i) = \langle B_i(x), \eta_i(y,v_i)\rangle$. Then Corollary 3.8 follows from Theorem 3.2. □

4 Systems of simultaneous equilibrium problems

As applications of Theorem 3.3, we have the following systems of simultaneous equilibrium problems.

Theorem 4.1 *Let* I, X_i, X, Y_i, Y, E_i, V_i *and* Z_i *be the same as in Theorem 3.1. For each* $i \in I$, *suppose that*

(i) S_i: $X \multimap X_i$ *is a compact continuous multivalued map with nonempty closed convex values;*
(ii) T_i: $X \multimap Y_i$ *is a compact continuous multivalued map with nonempty closed convex values;*
(iii) C_i: $X \multimap Z_i$ *and* $D_i : X \multimap Z_i$ *are closed multivalued maps with nonempty values;*
(iv) G_i: $X \times Y \times Y_i \multimap Z_i$ *is an u.s.c. multivalued map with nonempty compact values and* $G_i(x,y,y_i) \cap D_i(x) \neq \emptyset$ *and* F_i: $X \times Y \times X_i \multimap Z_i$ *is an u.s.c. multivalued map with nonempty compact values and* $F_i(x,y,x_i) \cap C_i(x) \neq \emptyset$ *for each* $x = (x_i)_{i \in I} \in X$, $y = (y_i)_{i \in I} \in Y$;
(v) *for each* $(x,v_i) \in X \times Y_i$, $y \multimap G_i(x,y,v_i)$ *is concave or* $\{0\}$*-quasiconvex; for each* $(y,u_i) \in Y \times X_i$, $w \multimap F_i(w,y,u_i)$ *is concave or* $\{0\}$*-quasiconvex;*
(vi) *for each* $(x,y) \in X \times Y$, $v_i \multimap G_i(x,y,v_i)$ *and* $u_i \multimap F_i(x,y,u_i)$ *are* $\{0\}$*-quasiconvex-like.*

Then there exists $(\bar{x},\bar{y}) \in X \times Y$ *such that* $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $F_i(\bar{x},\bar{y},u_i) \cap C_i(\bar{x}) \neq \emptyset$ *and* $G_i(\bar{x},\bar{y},v_i) \cap D_i(\bar{x}) \neq \emptyset$ *for all* $u_i \in S_i(\bar{x})$, $v_i \in T_i(\bar{x})$ *and all* $i \in I$.

Proof As in Theorem 3.2, we see $(x,y,u_i) \multimap -C_i(x) + F_i(x,y,u_i)$ and $(x,y,v_i) \multimap -D_i(x)+G_i(x,y,v_i)$ are closed multivalued maps. Then by Theorem 3.3 that there exist $\bar{x} = (\bar{x}_i)_{i \in I} \in X$, $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $0 \in -C_i(\bar{x})+F_i(\bar{x},\bar{y},u_i)$, and $0 \in -D_i(\bar{x}) + G_i(\bar{x},\bar{y},v_i)$ for all $u_i \in S_i(\bar{x})$, $v_i \in T_i(\bar{x})$ and all $i \in I$. Therefore, $F_i(\bar{x},\bar{y},u_i) \cap C_i(\bar{x}) \neq \emptyset$ and $G_i(\bar{x},\bar{y},v_i) \cap D_i(\bar{x}) \neq \emptyset$ for all $u_i \in S_i(\bar{x})$, $v_i \in T_i(\bar{x})$ and all $i \in I$. □

Following the same arguments as in Theorem 4.1, we have the following theorems.

Theorem 4.2 *In Theorem 4.1, if condition* (iii) *and* (iv) *are replaced by* (iii$'$) *and* (iv$'$), *respectively, where*

(iii$'$) C_i: $X \multimap Z_i$ *is a multivalued map such that* $\mathrm{int}C_i(x)$ *is nonempty for each* $x \in X$, W_i: $X \multimap Z_i$ *with* $W_i(x) := Z_i \setminus (-\mathrm{int}C_i(x))$ *and* D_i: $X \multimap Z_i$ *are closed multivalued maps with nonempty values;*

(iv$'$) G_i: $X \times Y \times Y_i \multimap Z_i$ *and* F_i: $X \times Y \times X_i \multimap Z_i$ *are u.s.c. multivalued maps with nonempty compact values and* $G_i(x,y,y_i) \cap D_i(x) \neq \emptyset$ *and* $F_i(x,y,x_i) \nsubseteq -\mathrm{int}C_i(x)$ *for each* $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$.

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ *such that* $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $F_i(\bar{x},\bar{y},u_i) \nsubseteq -\mathrm{int}C_i(\bar{x})$ *and* $G_i(\bar{x},\bar{y},v_i) \cap D_i(\bar{x}) \neq \emptyset$ *for all* $u_i \in S_i(\bar{x})$, $v_i \in T_i(\bar{x})$ *and all* $i \in I$.

Theorem 4.3 *In Theorem 4.1, if condition* (iii) *and* (iv) *are replaced by* (iii$'$) *and* (iv$'$), *respectively, where*

(iii$'$) D_i: $X \multimap Z_i$ *is a multivalued map such that* $\mathrm{int}D_i(x)$ *is nonempty for each* $x \in X$, $x \multimap Z_i \setminus (-\mathrm{int}D_i(x))$ *and* C_i: $X \multimap Z_i$ *are closed multivalued maps with nonempty values;*

(iv$'$) G_i: $X \times Y \times Y_i \multimap Z_i$ *and* F_i: $X \times Y \times X_i \multimap Z_i$ *are u.s.c. multivalued maps with nonempty compact values and* $G_i(x,y,y_i) \nsubseteq -\mathrm{int}D_i(x)$ *and* $F_i(x,y,x_i) \cap C_i(x) \neq \emptyset$ *for each* $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$.

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ *such that* $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$ *and* $F_i(\bar{x},\bar{y},u_i) \cap C_i(\bar{x}) \neq \emptyset$ *and* $G_i(\bar{x},\bar{y},v_i) \nsubseteq -\mathrm{int}D_i(\bar{x})$ *for all* $u_i \in S_i(\bar{x})$, $v_i \in T_i(\bar{x})$ *and all* $i \in I$.

Theorem 4.4 *In Theorem 4.1, if condition* (iii) *and* (iv) *are replaced by* (iii$'$) *and* (iv$'$), *respectively, where*

(iii$'$) C_i, D_i: $X \multimap Z_i$ *are multivalued maps such that* $\mathrm{int}C_i(x) \neq \emptyset$ *and* $\mathrm{int}D_i(x)$ *for each* $x \in X$, $x \multimap Z_i \setminus (-\mathrm{int}C_i(x))$ *and* $x \multimap Z_i \setminus (-\mathrm{int}D_i(x))$ *are closed multivalued maps;*

(iv$'$) G_i: $X \times Y \times Y_i \multimap Z_i$ *and* $F_i : X \times Y \times X_i \multimap Z_i$ *are u.s.c. multivalued maps with nonempty compact values and* $G_i(x,y,y_i) \nsubseteq -\mathrm{int}D_i(x)$ *and* $F_i(x,y,x_i) \nsubseteq -\mathrm{int}C_i(x)$ *for each* $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$.

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ *such that* $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$ *and* $F_i(\bar{x},\bar{y},u_i) \nsubseteq -\mathrm{int}C_i(\bar{x})$ *and* $G_i(\bar{x},\bar{y},v_i) \nsubseteq -\mathrm{int}D_i(\bar{x})$ *for all* $u_i \in S_i(\bar{x})$, $v_i \in T_i(\bar{x})$ *and all* $i \in I$.

Remark 4.1 If we put $F_i = 0$ for all $i \in I$ or $G_i = 0$ for all $i \in I$, then we obtain existence theorems of generalized vector quasiequilibrium problems.

5 Applications to optimization theory

In this section, we first establish the existence theorem of mathematical program with systems of variational inclusion constraints. From this result, we establish the existence theorems of mathematical programming with systems of equilibrium constraints, systems of bilevel problems and semi-infinite problems.

Theorem 5.1 *In Theorem 3.1. If X and Y are closed sets and condition* (i) *is replaced by*

(i′) S: $X \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values;*

and we suppose further that

(v) C_i: $X \multimap Z_i$ *is a closed multivalued map such that for each* $x \in X$, $C_i(x)$ *is a nonempty convex cone;*

(vi) F_i: $X \times Y \multimap Z_i$ *is a l.s.c. multivalued map with nonempty values and for each* $x \in X$, $y \multimap F_i(x, y)$ *is* $C_i(x)$*-quasiconcave-like;*

(vii) *for each* $x \in X$ *and* $y \in Y$, *there exists* $u_i \in S_i(x)$ *such that* $F_i(u, y) \subseteq C_i(x)$, *where* $u = (u_i)_{i\in I}$.

If h: $X \times Y \multimap Z_0$ *is an u.s.c. multivalued map with nonempty compact values, where* Z_0 *is a real t.v.s. ordered by a proper closed cone* D *in* Z_0, *then there exists an solution to the problem:*

$$\min_{(x,y)} h(x,y), x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \textit{ such that for each } i \in I, x_i \in S_i(x),$$
$$y_i \in T_i(x), F_i(x,y) \subseteq C_i(x) \textit{ and } 0 \in G_i(x,y,v_i) \textit{ for all } v_i \in T_i(x).$$

Proof For each $i \in I$, let A_i: $X \times Y \multimap X_i$ be defined by

$$A_i(x,y) = \{u_i \in S_i(x) \colon F_i(u,y) \subseteq C_i(x)\}.$$

By assumption, $A_i(x, y)$ is nonempty for each $x \in X$ and $y \in Y$. A_i is closed. Indeed, if $(x, y, u_i) \in \overline{GrA_i}$, then there exists a net $\{(x^\alpha, y^\alpha, u_i^\alpha)\}$ in GrA_i such that $(x^\alpha, y^\alpha, u_i^\alpha) \to (x, y, u_i)$. Let $u^\alpha = (u_i^\alpha)_{i\in I}$ and $u = (u_i)_{i\in I}$. One has $u_i^\alpha \in S_i(x^\alpha)$ and $F_i(u^\alpha, y^\alpha) \subseteq C_i(x^\alpha)$. By assumption and Theorem 2.1 that S_i is closed and $u_i \in S_i(x)$. Let $z_i \in F_i(u, y)$. Since F_i is l.s.c., there exists a net $z_i^\alpha \in F_i(u^\alpha, y^\alpha)$ such that $z_i^\alpha \to z_i$. We see that $z_i^\alpha \in C_i(x^\alpha)$. By assumption, C_i is closed, $z_i \in C_i(x)$. Hence $F_i(x, y) \subseteq C_i(x)$. Therefore $(x, y, u_i) \in GrA_i$ and GrA_i is closed. This shows that A_i is closed. It is easy to see that $A_i(x, y)$ is a closed set for each $x \in X$ and $y \in Y$. Since $A_i(X \times Y) \subseteq \overline{S_i(X)}$ and $\overline{S_i(X)}$ is compact, it follows from Theorem 2.1 that A_i: $X \times Y \multimap X_i$ is a compact u.s.c. multivalued map with nonempty closed values. $A_i(x, y)$ is convex for each $x \in X$, $y \in Y$ and $i \in I$. Indeed, let $u_i^1, u_i^2 \in A_i(x, y)$, $\lambda \in (0, 1)$, $u^1 = (u_i^1)_{i\in I}$, $u^2 = (u_i^2)_{i\in I}$, then $u_i^1, u_i^2 \in S_i(x)$, $F_i(u^1, y) \subseteq C_i(x)$ and $F_i(u^2, y) \subseteq C_i(x)$. By assumption, either

$$F_i(\lambda u^1 + (1-\lambda)u^2, y) \subseteq F_i(u^1, y) + C_i(x) \subseteq C_i(x) + C_i(x) \subseteq C_i(x)$$

or

$$F_i(\lambda u^1 + (1-\lambda)u^2, y) \subseteq F_i(u^2, y) + C_i(x) \subseteq C_i(x).$$

Since $S_i(x)$ is convex for each $x \in X$, $\lambda u_i^1 + (1-\lambda)u_i^2 \in S_i(x)$ and $\lambda u_i^1 + (1-\lambda)u_i^2 \in A_i(x, y)$ for each $(x, y) \in X \times Y$. Therefore A_i: $X \times Y \multimap X_i$ is a compact u.s.c. multivalued map with nonempty closed convex values. Then by Theorem 3.1 that there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} = (\bar{x}_i)_{i\in I}$, $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. Hence there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} = (\bar{x}_i)_{i\in I}$, $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $F_i(\bar{x}, \bar{y}) \subseteq C_i(\bar{x})$ and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. For each $i \in I$, let

$$M_i = \{(x,y) \in X \times Y : x = (x_i)_{i\in I}, y = (y_i)_{i\in I}, x_i \in S_i(x), y_i \in T_i(x),$$
$$F_i(x,y) \subseteq C_i(x) \text{ and } 0 \in G_i(x,y,v_i) \text{ for all } v_i \in T_i(x)\}$$

and $M = \cap_{i \in I} M_i$. Then $(\bar{x}, \bar{y}) \in M$ and $M \neq \emptyset$. We see that

$$M_i = \{(x,y) \in X \times Y\colon x = (x_i)_{i \in I}, y = (y_i)_{i \in I}, x_i \in S_i(x), F_i(x,y) \subseteq C_i(x) \text{ and } y_i \in H_i(x)\},$$

where H_i is defined as in Theorem 3.1. M_i is closed for each $i \in I$. Indeed, if $(x, y) \in \overline{M_i}$, then there exists a net $\{(x^\alpha, y^\alpha)\}$ in M_i such that $(x^\alpha, y^\alpha) \to (x, y)$. Let $x^\alpha = (x_i^\alpha)_{i \in I}$ and $y^\alpha = (y_i^\alpha)_{i \in I}$. One has $x_i^\alpha \in S_i(x^\alpha)$, $F_i(x^\alpha, y^\alpha) \subseteq C_i(x^\alpha)$ and $y_i^\alpha \in H_i(x^\alpha)$. Since S_i is u.s.c. multivalued map with nonempty closed values, S_i is closed. We see in Theorem 3.1 that H_i is closed. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$. Since x and Y are closed sets, $(x, y) \in X \times Y$. We also have $x_i \in S_i(x)$ and $y_i \in H_i(x)$. We prove $F_i(x, y) \subseteq C_i(x)$ in the first part of this theorem. Therefore $(x, y) \in M_i$ and M_i is closed for each $i \in I$. Hence $M = \cap_{i \in I} M_i$ is closed. Note that

$$M \subseteq \left[\prod_{i \in I} \overline{S_i(X)}\right] \times \left[\prod_{i \in I} \overline{T_i(X)}\right].$$

By assumption, $\overline{S_i(X)}$ and $\overline{T_i(X)}$ are compact, it follows from Lemma 3 [12] that $\left[\prod_{i \in I} \overline{S_i(X)}\right] \times \left[\prod_{i \in I} \overline{T_i(X)}\right]$ is compact. Therefore M is compact. Since $h : X \times Y \multimap Z_0$ is an u.s.c. multivalued map with nonempty compact values, it follows from Theorem 2.1 that $h(M)$ is compact. Then by Lemma 2.2 that $\min_D h(M) \neq \emptyset$. That is there exists a solution to the problem:

$$\min_{(x,y)} h(x,y), x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \text{ such that for each } i \in I, x_i \in S_i(x),$$
$$y_i \in T_i(x), F_i(x,y) \subseteq C_i(x) \quad \text{and} \quad 0 \in G_i(x, y, v_i) \quad \text{for all } v_i \in T_i(x). \qquad \square$$

Theorem 5.2 *In Theorem 5.1, if we assume that* $h\colon X \times Y \to \mathbb{R}$ *is an l.s.c. function, then there exists a solution to the problem:*

> $\min_{(x,y)} h(x,y)$, $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I}$, *such that for each* $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \subseteq C_i(x)$ *and* $0 \in G_i(x, y, v_i)$ *for all* $v_i \in T_i(x)$.

Proof Since $h : X \times Y \to \mathbb{R}$ is l.s.c. and M is compact, there exists $(\bar{x}, \bar{y}) \in M$ such that $h(\bar{x}, \bar{y}) = \min h(M)$. This completes the proof. $\square$

If we assume further conditions on Theorem 5.1, we have the following existence theorems of mathematical program with system of equilibrium constraints.

Theorem 5.3 *Let X, Y, conditions* (i′), (v), (vi) *and* (vii) *be the same as in Theorem 5.1, if we assume further that*

(viii) $D_i\colon X \multimap Z_i$ *is a closed multivalued map and* $G_i\colon X \times Y \times Y_i \multimap Z_i$ *is an u.s.c. multivalued map with nonempty compact values;*

(viiii) *for each* $(x, v_i) \in X \times Y_i$, $y \multimap G_i(x, y, v_i)$ *is concave or* $\{0\}$*-quasiconvex, and for each* $(x, y) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ *is* $\{0\}$*-quasiconvex-like and* $G_i(x, y, y_i) \cap D_i(x) \neq \emptyset$, *where* $y = (y_i)_{i \in I}$.

(x) $T_i\colon X \multimap Y_i$ *is a compact continuous multivalued map with nonempty closed convex values.*

Then there exists a solution to the problem:
$\min_{(x,y)} h(x,y)$, $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I}$ *such that for each* $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x,y) \subseteq C_i(x)$ *and* $G_i(x, y, v_i) \cap D_i(x) \neq \emptyset$ *for all* $v_i \in T_i(x)$.

Proof Let A_i be defined as in Theorem 5.1. We show in Theorem 5.1 that $A_i : X\times Y \multimap X_i$ is compact u.s.c. multivalued map with compact values. It follows from Corollary 3.1 and Theorem 5.1 that there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} = (\bar{x}_i)_{i\in I}$ and $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $F_i(\bar{x}, \bar{y}) \subseteq C_i(\bar{x})$ and $G_i(\bar{x}, \bar{y}, v_i) \cap D_i(\bar{x}) \neq \emptyset$ for all $v_i \in T_i(\bar{x})$. For each $i \in I$, let

$$M_i = \{(x, y) \in X \times Y : x = (x_i)_{i\in I}, y = (y_i)_{i\in I}, x_i \in S_i(x), y_i \in T_i(x),$$
$$F_i(x, y) \subseteq C_i(x) \text{ and } G_i(x, y, v_i) \cap D_i(x) \neq \emptyset \text{ for all } v_i \in T_i(x)\}.$$

M_i is closed for each $i \in I$. Indeed, if $(x, y) \in \overline{M_i}$, then there exists a net $\{(x^\alpha, y^\alpha)\}_{\alpha\in\Lambda}$ in M_i such that $(x^\alpha, y^\alpha) \to (x, y)$. Let $x^\alpha = (x_i^\alpha)_{i\in I}$ and $y^\alpha = (y_i^\alpha)_{i\in I}$. One has $x_i^\alpha \in S_i(x^\alpha)$, $y_i^\alpha \in T_i(x^\alpha)$, $F_i(x^\alpha, y^\alpha) \subseteq C_i(x^\alpha)$ and $G_i(x^\alpha, y^\alpha, v_i) \cap D_i(x^\alpha) \neq \emptyset$ for all $v_i \in T_i(x^\alpha)$. Let $v_i \in T_i(x)$, then there exists a net $\{v_i^\alpha\}_{\alpha\in\Lambda}$, $v_i^\alpha \in T_i(x^\alpha)$ for all $\alpha \in \Lambda$ such that $v_i^\alpha \to v_i$. Let $u_i^\alpha \in G_i(x^\alpha, y^\alpha, v_i^\alpha) \cap D_i(x^\alpha)$. Then $u_i^\alpha \in G_i(x^\alpha, y^\alpha, v_i^\alpha)$ and $u_i^\alpha \in D_i(x^\alpha)$. Let $A = \{x^\alpha : \alpha \in \Lambda\} \cup \{x\}$, $B = \{y^\alpha : \alpha \in \Lambda\}$, $L = \{v_i^\alpha : \alpha \in \Lambda\} \cup \{v_i\}$. Then A, B, C are compact sets.

Since G_i is an u.s.c. multivalued map with nonempty compact values, it follows from Theorem 2.1, $G_i(A \times B \times C)$ is a compact set and $\{u_i^\alpha\}$ has s subnet $\{u_i^{\alpha_\lambda}\}$ in $G_i(A\times B\times B\times C)$ such that $u_i^{\alpha_\lambda} \to u_i$. By Theorem 2.1, G_i is closed, and $u_i \in G_i(x, y, v_i)$. By assumption, D_i is closed and $u_i \in D_i(x)$. As before, we see that $x_i \in S_i(x)$, $y_i \in T_i(x)$ and $F_i(x, y) \subseteq C_i(x)$. This shows that M_i is a closed set. Let $M = \cap_{i\in I} M_i$. Then we follow the same argument as in Theorem 5.1, we can prove Theorem 5.3. □

Following the same argument as in Theorem 5.3, we have the following theorem.

Theorem 5.4 *In Theorem 5.3, if conditions* (viii) *and* (viiii) *are replaced by* (iii′) *and* (ix′)*, respectively, where*

- (viii′) *D_i: $X \multimap Z_i$ is a multivalued map such that* $\operatorname{int} D_i(x) \neq \emptyset$ *for each $x \in X$ and $W_i : X \multimap Z_i$ which is defined by $W_i(x) = Z_i \setminus (-\operatorname{int} D_i(x))$ is a closed multivalued map and G_i: $X \times Y \times Y_i \multimap Z_i$ is an u.s.c. multivalued map with nonempty compact values;*
- (ix′) *for each $(x, v_i) \in X \times Y_i$, $y \multimap G_i(x, y, v_i)$ is concave or $\{0\}$-quasiconvex and for each $(x, y) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and*

$$G_i(x, y, y_i) \nsubseteq -\operatorname{int} D_i(x), \quad \text{where } y = (y_i)_{i\in I}.$$

Then there exists a solution to the problem:

$\min_{(x,y)} h(x, y)$, *$x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I}$ such that for each $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x, y) \subseteq C_i(x)$ and $G_i(x, y, v_i) \nsubseteq -\operatorname{int} D_i(x)$ for all $v_i \in T_i(x)$.*

If $Z_i = \mathbb{R}$ for all $i \in I$, we have the following theorem.

Theorem 5.5 *Let $h : X \times Y \to \mathbb{R}$ be a l.s.c. function. In Theorem 3.1, if we assume condition (i) is replaced by (i′), and conditions* (ii) *and* (iii) *are replaced by* (iv)$_9$*, where*

- (i′) *$S_i : X \to X_i$ is a compact u.s.c. multivalued map with nonempty closed convex values;*
- (iv)$_9$ *$G_i : X \times Y \times Y_i \multimap \mathbb{R}$ is a continuous multivalued map with nonempty compact values such that for each $(x, v_i) \in X \times Y_i$, $y \multimap G_i(x, y, v_i)$ is concave or $\{0\}$-quasiconvex and for each $(x, y) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ is $\mathbb{R}^+$-quasiconvex-like and $G_i(x, y, y_i) \cap \mathbb{R}^+ \neq \emptyset$.*

Suppose further that (iii)$_9$ F_i: $X \times Y \multimap \mathbb{R}$ is a l.s.c. multivalued map with nonempty values and for each $x \in X$, $y \multimap F_i(x,y)$ is $\mathbb{R}_+$-quasiconcave-like.

Then there exists a solution to the problem:

$$\min_{(x,y)} h(x,y), x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \text{ such that for each } i \in I,$$
$$x_i \in S_i(x), y_i \in T_i(x), F_i(x,y) \subseteq \mathbb{R}^+ \text{ and } G_i(x,y,v_i) \cap \mathbb{R}^+ \neq \emptyset \text{ for all } v_i \in T_i(x).$$

Proof If we let $C_i(x) = \mathbb{R}^+$ and $D_i(x) = \mathbb{R}^+$ for all $x \in X$ and for all $i \in I$. It is easy to see that if for each $(x,y) \in X \times Y$, $v_i \multimap G_i(x,y,v_i)$ is $\mathbb{R}^+$-quasiconvex-like, then for each $(x,y) \in X \times Y$, $v_i \multimap -\mathbb{R}^+ + G_i(x,y,v_i)$ is $\{0\}$-quasiconvex-like. We also see that

$$G_i(x,y,y_i) \cap \mathbb{R}^+ \neq \emptyset \Leftrightarrow 0 \in -\mathbb{R}^+ + G_i(x,y,y_i).$$

We follow the first part of Theorem 5.3 that there exists $(\bar{x},\bar{y}) \in X \times Y$, $\bar{x} = (\bar{x}_i)_{i\in I}$, $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x}), F_i(\bar{x},\bar{y}) \subseteq \mathbb{R}^+$ and $G_i(\bar{x},\bar{y},v_i) \cap \mathbb{R}^+ \neq \emptyset$ for all $v_i \in T_i(\bar{x})$. For each $i \in I$, let

$$M_i = \{(x,y) \in X \times Y : x = (x_i)_{i\in I}, y = (y_i)_{i\in I}, x_i \in S_i(x), y_i \in T_i(x),$$
$$F_i(x,y) \subseteq \mathbb{R}^+ \text{ and } G_i(x,y,v_i) \cap \mathbb{R}^+ \neq \emptyset \text{ for all } v_i \in T_i(x)\}$$

and $M = \cap_{i\in I} M_i$. Then M is compact. Since h is l.s.c. on M, there exists $(\bar{x},\bar{y}) \in X \times Y$ such that $h(\bar{x},\bar{y}) = \min h(M)$. Theorem 5.5 follows. □

Corollary 5.1 *In Theorem 5.5, if conditions* (iii)$_9$ *and* (iv)$_9$ *are replaced by* (iii)$_{10}$ *and* (iv)$_{10}$*, respectively, where*

(iii)$_{10}$ *F_i: $X \times Y \to R$ is a continuous function such that for each $x_i \in X_i$, $y \to F_i(x,y)$ is quasiconcave;*
(iv)$_{10}$ *$G_i : X \times Y \times Y_i \to \mathbb{R}$ is a continuous function such that for each $(x,v_i) \in X \times Y_i$, $y \to G_i(x,y,v_i)$ is affine or $\{0\}$-quasiconvex and for each $(x,y) \in X \times Y$, $v_i \to G_i(x,y,v_i)$ is quasiconvex and $G_i(x,y,y_i) \geq 0$.*

Then there exists a solution to the problem:

$$\min_{(x,y)} h(x,y), x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \text{ such that for each } i \in I, x_i \in S_i(x),$$
$$y_i \in T_i(x), F_i(x,y) \geq 0 \quad \text{and} \quad G_i(x,y,v_i) \geq 0 \quad \text{for all } v_i \in T_i(x).$$

For another special case of Corollary 5.1, we have the following existence theorem of bilevel problem.

Corollary 5.2 *In Corollary 5.1, if condition* (iv)$_9$ *is replaced by (a), where*

(a) *$Q_i : X \times Y_i \to \mathbb{R}$ is a continuous function such that for each $x \in X$, $y_i \to Q_i(x,y_i)$ is affine or $\{0\}$-quasiconvex.*

Then there exists a solution to the problem:

$$\min_{(x,y)} h(x,y), x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \text{ such that for each } i \in I, x_i \in S_i(x),$$
$$y_i \in T_i(x), F_i(x,y) \geq 0 \text{ and } y_i \text{ is a solution to the problem: } \min_{v_i \in T_i(x)} Q_i(x,v_i).$$

Proof Let $G_i(x,y,v_i) = Q_i(x,v_i) - Q_i(x,y_i)$ for all $i \in I$ and for $y = (y_i)_{i\in I}$. By assumption, for each $x \in X$, $v_i \to Q_i(x,v_i)$ is affine or $\{0\}$-quasiconvex, it is easy to see that for each $(x,v_i) \in X \times Y_i$, $y \to G_i(x,y,v_i)$ is affine or $\{0\}$-quasiconvex and for each $(x,y) \in X \times Y$, $v_i \to G_i(x,y,v_i)$ is quasiconvex. Then by Corollary 5.1 that there exists a solution to the problem:

$\min_{(x,y)} h(x,y), x = (x_i)_{i\in I}, y = (y_i)_{i\in I}$, such that for each $i \in I, x_i \in S_i(x), y_i \in T_i(x)$, $F_i(x,y) \geq 0$ and $G_i(x,y,v_i) \geq 0$ for all $v_i \in T_i(x)$.

That is, y_i is a solution to the problem: $\min_{v_i \in T_i(x)} Q_i(x,v_i)$. □

Remark 5.1 In Corollary 5.2, if we assume further that $Q_i(x,y_i) \geq 0$ for $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I}$ with $x_i \in S_i(x)$ and $y_i \in T_i(x)$, then there exists a solution to the semi-infinite problem with systems of equilibrium constraints:

$$\min_{(x,y)} h(x,y), x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \text{ such that for each } i \in I, x_i \in S_i(x),$$
$$y_i \in T_i(x), F_i(x,y) \geq 0 \text{ and } Q_i(x,v_i) \geq 0 \text{ for all } v_i \in T_i(x).$$

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